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ORIGINAL PAPER

The influence of the support functions on the quality of enhanced multivariance product representation

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Abstract This paper presents recently developed Enhanced Multivariance Product Representation (EMPR) method for multivariate functions. EMPR disintegrates a multivariate function to components which are respectively constant, univariate, bivariate and so on in ascending multivariance. Although the EMPR method has the same philosophy with the High Dimensional Model Representation (HDMR) method, it has been proposed to get better quality than HDMR's with the help of the support functions. For this purpose, we investigate the EMPR truncation qualities with respect to the selection of the support functions. The obtained results and a number of numerical implementations to show the efficiency of the method are also given in this paper.

Keywords High dimensional model representation · Multivariate functions · Approximation

1 Introduction

Multivariance and multivariate functions cause numerical and computational disadvantages in scientific problems. The increasing tendency is to deal with less-variate functions in multivariate problems (In many problems of chemistry, this has great importance also). One way is to use divide-and-conquer algorithms for the needed solutions. High Dimensional Model Representation (HDMR) is a divide-and-conquer method and was first proposed by Sobol in 1993 [1]. This method gives the chance of dealing with less-variate functions instead of multivariate functions in scientific

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problems. Dealing with less-variate functions in multivariate problems decrease the mathematical and computational complexities.

HDMR, was initially developed by Sobol [1] with unit weight and the unit interval (0,1). Rabitz [2–4] and his group extended the HDMR definition by using different intervals and weights. In the same period Demiralp [5–10] and his group, in addition to the improvement of the method, they developed also new HDMR based algorithms for various engineering problems.

HDMR is an expansion including, first a constant, then univariate functions followed by bivariate functions and a certain number of higher variate functions. The philosophy of the HDMR method is to represent the given multivariate function in terms of these mentioned less-variate functions. The main purpose is to determine the structures of these HDMR components and the total number of these components is 2^N. This means that we need 2^N components to be determined to exactly represent a given multivariate function. Although it is finite the number of terms grow exponentially as N increases unboundedly and we may need to deal with zillion number of terms. Instead of taking all components, we can deal with a small number of HDMR components to represent our multivariate function by using less-variate functions. That is, we can truncate the HDMR expansion at some level and obtain an approximate representation for the given multivariate function. This type of truncation gives us HDMR approximants. These approximants may be used instead of a given multivariate function. This means that the truncated HDMR is also an approximation method. Hence, the measuring approximation quality of the expansion becomes very important as a result of the truncation philosophy. For these purposes, some quality measurers are defined in the HDMR method. Detailed information about these definitions and the general algorithm for the HDMR method are given in the next section.

The expansion of the HDMR method is a finite sum. As this expansion has an additive structure we have to emphasize an important point that the HDMR approximants work best for the multivariate functions having purely additive natures.

If the multivariate function under consideration has a multiplicative nature then the HDMR method becomes unemployable. Another method is needed to obtain acceptable approximate representations for the multivariate problems. The other method is called Factorized HDMR (FHDMR) [9,10] which has a product type expansion based on the HDMR components and better representations rather than the HDMR method are obtained through this FHDMR method.

In FHDMR, we cannot define monotonously growing truncation quality measurers like we define in the HDMR method. Additionally, to get the desired quality through FHDMR all HDMR components are needed to be retained. This increases the numerical and the computational complexity of the given multivariate problems. Hence, we need another method in order to construct an approximate represantation for the given problems in which the multivariate function under consideration has a product type nature. This new method is called Enhanced Multivariance Product Representation (EMPR) [11,12]. This new method has an expansion which is very similar to the HDMR expansion. There exists some extra univariate support functions in this method. Defining these support functions provides us flexibility to determine better HDMR approximants for the product type multivariate functions and to use the quality measurers of the HDMR method which cannot be defined within the same philosophy of HDMR, in FHDMR method to measure the quality of the obtained approximant. If these univariate support functions are selected as unit constant function, 1, the EMPR method becomes equivalent to the HDMR method.

The paper is organized as follows. Second section involves a brief overview of HDMR method. Third section covers the Enhanced Multivariance Product Representation (EMPR) method. The fourth section includes implementations to measure the efficiency of the EMPR method with the help of MuPAD [13], Computer Algebra System, to have any desired precision in calculations and to use its symbolic programming nature. Finally, the fifth section presents concluding remarks.

2 Plain high dimensional model representation (Plain HDMR)

The basic equation of HDMR for a given multivariate function, $f(x_1, \ldots, x_N)$, is as

$$f(x_1, \ldots, x_N) = f_0 + \sum_{i_1=1}^{N} f_{i_1}(x_{i_1}) + \sum_{\substack{i_1, i_2=1 \\ i_1 < i_2}}^{N} f_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + f_{12\ldots N}(x_1, \ldots, x_N) \tag{1}$$

where N stands for the number of the independent variables. This expansion is a finite sum and has 2^N components on the right hand side which are mutually orthogonal [10]. If this expansion is truncated at some level this exact expansion becomes an approximation to represent the target multivariate function $f(x_1, \ldots, x_N)$. The right hand side terms of expansion are called constant, univariate, bivariate components and so on respectively.

Orthogonal feature is defined through the inner product of the Hilbert space spanned by the square integrable functions over the rectangular hyperprism defined as the cartesian product of the intervals $a_i \le x_i \le b_i$. The inner product mentioned here is explicitly given below

$$(u, v) \equiv \int_{a_1}^{b_1} dx_1 \ldots \int_{a_N}^{b_N} dx_N W(x_1, \ldots, x_N) u(x_1, \ldots, x_N) v(x_1, \ldots, x_N) \tag{2}$$

where $u(x_1, \ldots, x_N)$ and $v(x_1, \ldots, x_N)$ are two arbitrary square integrable multivariate functions and, for consistency, the weight function appearing here is assumed to be a product type function as follows

$$W(x_1, \ldots, x_N) \equiv \prod_{i=1}^{N} W_i(x_i), \quad x_i \in [a_i, b_i], \quad 1 \le i \le N \tag{3}$$

The HDMR components at the right hand side of Eq. (1) can be determined uniquely by using the vanishing conditions or the corresponding orthogonality conditions given below respectively.

$$\int_{a_{i_s}}^{b_{i_s}} dx_{i_s} W_{i_s}(x_{i_s}) f_{i_1 \ldots i_k}(x_{i_1}, \ldots, x_{i_k}) = 0, \quad i_1 \leq i_s \leq i_k \tag{4}$$

$$(f_{i_1 i_2 \ldots i_k}, f_{i_1 i_2 \ldots i_l}) = 0, \ \{i_1, i_2, \ldots, i_k\} \not\equiv \{i_1, i_2, \ldots, i_l\}, \ 1 \leq k, l \leq N \tag{5}$$

The following normalizations of weight factors are assumed to facilitate further algebraic manipulations on HDMR construction.

$$\int_{a_i}^{b_i} dx_i W_i(x_i) = 1, \quad 1 \leq i \leq N \tag{6}$$

Now, it is time to construct the algorithm of finding the general structures of the HDMR components. First step is to define a projection operator for the determination of the constant HDMR term. This operator which maps from the entire space to the subspace of constant functions, can be defined over a square integrable function, $g(x_1, \ldots, x_N)$ as

$$\mathscr{P}_0 g(x_1, \ldots, x_N) \equiv \int_{a_1}^{b_1} dx_1 \ldots \int_{a_N}^{b_N} dx_N W(x_1, \ldots, x_N) g(x_1, \ldots, x_N) \tag{7}$$

which is applied to both sides of Eq. (1) and produces the following result if vanishing conditions and the normalized weight function factors are used.

$$f_0 = \mathscr{P}_0 f(x_1, \ldots, x_N) \equiv \int_{a_1}^{b_1} dx_1 \ldots \int_{a_N}^{b_N} dx_N W(x_1, \ldots, x_N) f(x_1, \ldots, x_N) \tag{8}$$

Next step is to define another projection operator, which maps from the whole space to its subspace of univariate functions, to obtain the general structure for the univariate HDMR components again by using a square integrable function, $g(x_1, \ldots, x_N)$ as

$$\mathscr{P}_i g(x_1, \ldots, x_N) \equiv \int_{a_1}^{b_1} dx_1 W_1(x_1) \ldots \int_{a_{i-1}}^{b_{i-1}} dx_{i-1} W_{i-1}(x_{i-1}) \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} W_{i+1}(x_{i+1})$$
$$\times \ldots \times \int_{a_N}^{b_N} dx_N W_N(x_N) g(x_1, \ldots, x_N), \quad 1 \leq i \leq N \tag{9}$$

If this operator is applied to the both sides of the HDMR equation given in (1), then the general structure for the univariate components is obtained as follows.

$$
\begin{aligned}
f_i(x_i) = \mathscr{P}_i f(x_1,\ldots,x_N) = & \int_{a_1}^{b_1} dx_1 W_1(x_1)\ldots\int_{a_{i-1}}^{b_{i-1}} dx_{i-1} W_{i-1}(x_{i-1}) \\
& \times \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} W_{i+1}(x_{i+1}) \\
& \times\cdots\times \int_{a_N}^{b_N} dx_N W_N(x_N) f(x_1,\ldots,x_N) - f_0 \\
& 1 \le i \le N
\end{aligned}
\qquad (10)
$$

The remaining HDMR components can be determined in the same manner, that is, a projection operator can be defined for each HDMR component and then that operator can be applied to both sides of the HDMR equation. For simplicity, we do not give the details of this procedure as it is clear to construct the algorithm including these definitions.

As there exist 2^N components in the HDMR expansion, to obtain the structures of all these components is a very expensive task even in computers. This urges us to truncate HDMR at certain and rather low multivariances as long as the truncation has a good representation quality. The general tendency is to truncate, at most, at bivariance level. These truncations are called "HDMR approximants" and are explicitly given below

$$
\begin{aligned}
S_0^{(h)}(x_1,\ldots,x_N) &= f_0 \\
S_1^{(h)}(x_1,\ldots,x_N) &= S_0^{(h)}(x_1,\ldots,x_N) + \sum_{i=1}^{N} f_i(x_i) \\
&\;\vdots \\
S_k^{(h)}(x_1,\ldots,x_N) &= S_{k-1}(x_1,\ldots,x_N) + \sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\cdots<i_k}}^{N} f_{i_1\ldots i_k}(x_{i_1},\ldots,x_{i_k}) \\
& 1 \le k \le N
\end{aligned}
\qquad (11)
$$

The next important issue is to determine the qualities of these truncations. The following entities which are called "Additivity Measurers" were defined by Demiralp [10] to get insight about these qualities

$$
\sigma_0^{(h)} \equiv \frac{1}{\|f\|^2}\,\|f_0\|^2
$$

$$\begin{aligned}\sigma_1^{(h)} &\equiv \frac{1}{\|f\|^2}\sum_{i=1}^{N}\|f_i\|^2+\sigma_0^{(h)}\\ &\vdots\\ \sigma_N^{(h)} &\equiv \frac{1}{\|f\|^2}\|f_{12\ldots N}\|^2+\sigma_{N-1}^{(h)}\end{aligned} \tag{12}$$

which necessitate the square integrability of the original function beside the demand for laying in the Hilbert space defined on the inner product of HDMR. Here, in this equalities, $\sigma_0^{(h)}$ is called "Constancy Measurer" and it defines the contribution percentage of the constant term to the HDMR's norm square. $\sigma_1^{(h)}$ is called "First Order Additivity Measurer" and defines the contribution percentage of the constant term and univariate terms all together to the HDMR's norm square. As a generalization $\sigma_k^{(h)}$ is called "k-th Order Additivity Measurer" and defines the contribution percentage of the all terms from constant term to k-th order term inclusive to the HDMR's norm square. The superscript letter, h, corresponds to the word HDMR as we need to use the same symbol for the quality measurers of the EMPR method. These additivity measurers forms a well–ordered sequence. That is,

$$0 \le \sigma_0^{(h)} \le \cdots \le \sigma_N^{(h)} = 1 \tag{13}$$

The closer the $\sigma_k^{(h)}$ is to one, the better the quality of the k-th approximation.

3 Enhanced multivariance product representation (EMPR)

Enhanced Multivariance Product Representation which is somehow an extension to HDMR can be given as follows [11,12]

$$\begin{aligned}f(x_1,\ldots,x_N) = f_0\prod_{j=1}^{N}s_j(x_j)+\sum_{i=1}^{N}f_i(x_i)\prod_{\substack{j=1\\ j\neq i}}^{N}s_j(x_j)\\ +\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N}f_{i_1,i_2}(x_{i_1},x_{i_2})\prod_{\substack{j=1\\ j\neq i_1,i_2}}^{N}s_j(x_j)\\ +\cdots+f_{12\ldots N}(x_1,x_2,\ldots,x_N)\end{aligned} \tag{14}$$

where $s_j(x_j)$s, $(1 \le j \le N)$ are called "Support Functions". In this work, all of these support functions are chosen as univariate functions. Of course, this selection can be in a different way such as they may be bivariate functions or some other structures making the analysis harder and perhaps impractical in certain cases. On the other hand, it is clear that if these univariate support functions are selected as unit constant functions, the EMPR method becomes equivalent to the HDMR method.

The unknowns of this expansion are again constant, f_0; univariate terms, $f_i(x_i)$; bivariate terms, $f_{i_1,i_2}(x_{i_1}, x_{i_2})$; and so on. The main purpose of this method is again to determine the general structures of these terms like in HDMR method but this time by using the features of the above mentioned support functions. After the determination of these components the next step of the algorithm is to represent the given multivariate function in terms of these components by using the EMPR expansion given in (14).

EMPR uses an orthogonal hyperprismatic hypervolume and its similar type lower dimensional subvolumes in integrations for its definition. This means that the independent variable of EMPR vary in intervals with constant endpoints (they could be depending on the other variables like in triangles but do not). This separates the all integrations to univariate integrations analogously to HDMR.

Although they somehow play the role of weight functions, support functions need not to be true weight functions. Hence, their integrals may vanish in certain circumstances. This urges us to use square integration not sole integration for their normalizations. Hence from now on we assume that each support function's square integral over the relevant interval under the given weight is equal to 1.

The important point here is the selection method of these support functions. Because, the structure of the functions directly effect the approximation performance of the EMPR method. Not to interrupt the explanation of the EMPR algorithm's steps the support function selection process will be given in the ending part of this section.

Returning to the steps of our algorithm the coming part is to define the projection operators for the determination of the expansion's constant and univariate terms. For this purpose, the following operators are defined.

$$\mathscr{I}_0 f(x_1,\ldots,x_N) \equiv \int_{a_1}^{b_1} dx_1 W_1(x_1)\ldots\int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{j=1}^{N} s_j(x_j) f(x_1,\ldots,x_N) \tag{15}$$

$$\begin{aligned}\mathscr{I}_i f(x_1,\ldots,x_N) \equiv &\int_{a_1}^{b_1} dx_1 W_1(x_1)\ldots\int_{a_{i-1}}^{b_{i-1}} dx_{i-1} W_{i-1}(x_{i-1}) \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} W_{i+1}(x_{i+1})\ldots\\ &\times\int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{\substack{j=1\\ j\neq i}}^{N} s_j(x_j) f(x_1,\ldots,x_N)\end{aligned} \tag{16}$$

Now we impose the vanishing conditions on EMPR components, subindexed fs, such that the integral of each nonconstant component with respect to anyone of the independent variables in its argument, after multiplying it by the support function depending on that integration variable, under the corresponding weight factor vanishes. That is,

$$\int_{a_{i_\ell}}^{b_{i_\ell}} dx_{i_\ell} W_{i_\ell}(x_{i_\ell}) s_{i_\ell}(x_{i_\ell}) f_{i_1,\ldots,i_k}(x_{i_1},\ldots,x_{i_k}) = 0,$$

$$\int_{a_i}^{b_i} dx_i\, W_i(x_i) = 1, \qquad \int_{a_i}^{b_i} dx_i\, W_i(x_i) s_i(x_i)^2 = 1, \quad 1 \le i, k \le N \tag{17}$$

where i_ℓ can take values in the set $\{x_1, \ldots, x_k\}$. This permits us to get the constant EMPR component as follows.

$$f_0 = \int_{a_1}^{b_1} dx_1 W_1(x_1) \ldots \int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{j=1}^{N} s_j(x_j) f(x_1, \ldots, x_N) \tag{18}$$

Next step is to determine the general structures of the univariate EMPR components. For this purpose, the $\mathscr{I}_i$ operator given in (16) is applied to the both sides of the EMPR equality by taking the vanishing and other conditions into consideration. As a result, the following structure for the univariate EMPR components is obtained.

$$\begin{aligned} f_i(x_i) = &\int_{a_1}^{b_1} dx_1 W_1(x_1) \ldots \int_{a_{i-1}}^{b_{i-1}} dx_{i-1} W_{i-1}(x_{i-1}) \int_{a_{i+1}}^{b_{i+1}} dx_{i+1} W_{i+1}(x_{i+1}) \ldots \\ &\times \int_{a_N}^{b_N} dx_N W_N(x_N) \prod_{\substack{j=1 \\ j \ne i}}^{N} s_j(x_j) f(x_1, \ldots, x_N) - f_0 s_i(x_i) \end{aligned} \tag{19}$$

To reduce the computational complexity we deal with only univariate EMPR components at most. However, the higher variate EMPR terms can be determined by following the same philosophy.

The truncations in EMPR produce the following approximants. We call them “Constant” and “Univariate” EMPR Approximants when only constant term, and both constant and univariate terms, are retained.

$$\begin{aligned} S_0^{(e)}(x_1, \ldots, x_N) &= f_0 \prod_{j=1}^{N} s_j(x_j), \\ S_1^{(e)}(x_1, \ldots, x_N) &= S_0^{(e)}(x_1, \ldots, x_N) + \sum_{i=1}^{N} f_i(x_i) \prod_{\substack{j=1 \\ j \ne i}}^{N} s_j(x_j) \\ &\vdots \\ S_k^{(e)}(x_1, \ldots, x_N) &= S_{k-1}^{(e)}(x_1, \ldots, x_N) + \sum_{\substack{i_1, \ldots, i_k = 1 \\ i_1 < \cdots < i_k}}^{N} f_{i_1 \ldots i_k}(x_{i_1}, \ldots, x_{i_k}) \prod_{\substack{j=1 \\ j \ne i_1, \ldots, i_k}}^{N} s_j(x_j) \\ &1 \le k \le N \end{aligned} \tag{20}$$

The following "EMPR Truncation Quality Measurers" are used to measure the approximation qualities obtained by using these approximants.

$$\sigma_0^{(e)} \equiv \frac{1}{\|f\|^2} \left\| f_0 \prod_{j=1}^{N} s_j \right\|^2 ,$$

$$\sigma_1^{(e)} \equiv \frac{1}{\|f\|^2} \sum_{i=1}^{N} \left\| f_i \prod_{\substack{j=1 \\ j \neq i}}^{N} s_j \right\|^2 + \sigma_0^{(e)} \tag{21}$$

$$\vdots$$

$$\sigma_N^{(e)} \equiv \frac{1}{\|f\|^2} \|f_{12\ldots N}\|^2 + \sigma_{N-1}^{(e)}$$

These measurers are well ordered, that is, $0 \leq \sigma_0^{(e)} \leq \sigma_1^{(e)} \cdots \leq \sigma_N^{(e)} = 1$. This facilitates an error analysis for the approximations based on EMPR truncations.

The very important point here is to select the appropriate support functions in the algorithm to obtain successfull approximations for the given multivariate function. Of course, this selection process is not unique. One can construct another mechanism to determine other type of support functions for the given multivariate engineering problems. However, in this work our aim is to use the following definition to determine the efficient support function structures.

$$s_j(x_j) = \frac{\int_{a_1}^{b_1} dx_1 \ldots \int_{a_{j-1}}^{b_{j-1}} dx_{j-1} \int_{a_{j+1}}^{b_{j+1}} dx_{j+1} \ldots \int_{a_N}^{b_N} dx_N f(x_1, \ldots, x_N)}{[\int_{a_j}^{b_j} dx_j [\int_{a_1}^{b_1} dx_1 \ldots \int_{a_{j-1}}^{b_{j-1}} dx_{j-1} \int_{a_{j+1}}^{b_{j+1}} dx_{j+1} \ldots \int_{a_N}^{b_N} dx_N f(x_1, \ldots, x_N)]^2]^{\frac{1}{2}}} \tag{22}$$

This somehow reflects the unidirectional dependence of the target function on the relevant independent variable in the mean.

4 Numerical implementations

Several numerical implementations are given in this section to show the performance of our new method. As mentioned before, the support functions play very important role in the method. Hence, the structures and the effects of these functions are examined very carefully in our numerical implementations.

As a first example, a multivariate function having 5 independent variables is chosen as

$$f(x_1, x_2, x_3, x_4, x_5) = x_1 * x_2 * x_3 * x_4 * x_5 \tag{23}$$

which has a purely multiplicative nature. To represent this function in terms of the EMPR components we need to define the support functions appearing inside the EMPR expansion. When the relation given in (22) is used to define these support functions, the following structure is obtained for each one.

$$s_i(x_i) = 1.732050808 x_i, \quad 1 \le i \le 5 \tag{24}$$

As the testing function has 5 independent variables there must be 5 support functions each of which depends on a different independent variable.

After some calculations by using these support functions and the related steps of the EMPR method given in Sect. 3 the values for the zeroth order and the first order EMPR quality measurers are obtained as follows.

$$\begin{aligned} \sigma_0^{(e)} &= 1.0 \\ \sigma_1^{(e)} &= 1.0 \end{aligned} \tag{25}$$

If we use the HDMR method instead of our new method in this example, we would obtain the values for the constancy and the first order additivity measurers as $\sigma_0^{(h)} = 0.2373046875$ and $\sigma_1^{(h)} = 0.6328125$ respectively. Since we are dealing with a multiplicative nature the EMPR method represents the given function better than the HDMR method (in fact exactly for purely multiplicative functions). It is clear that the support functions make efficient corrections on the representation of the given multivariate function and even the constant EMPR component with the help of the support functions can exactly represent the given multivariate function.

The testing multivariate function for the second implementation is selected as

$$f(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4)^m \tag{26}$$

where it has again 4 independent variables and m corresponds to any positive integer. The integer value will arrange the multiplicativity level of the multivariate function. When m equals to 1, it means our function has a purely additive nature. When it takes higher values its nature becomes more multiplicative. This results in worse HDMR representations for the given function. As the multiplicativity level of the function to be represented by the HDMR method increases, HDMR fails and it is observed that the efficiency of the EMPR method increases rapidly.

For the first case, let us assume that m equals to 1 and determine the structure of each support function by using again relation (22).

$$s_i(x_i) = 0.4948716593 x_i + 0.742307489, \quad 1 \le i \le 4 \tag{27}$$

For the construction of different cases for this example, we choose the values 1, 2, 3, 4, 5, 6 and 7 for m. As the value of m increases, the multiplicativity dominance of the multivariate function increases. To represent each of them we need to define support

Table 1 Quality measurer values obtained in the second example for differerent m values

m	$\sigma_0^{(h)}$	$\sigma_0^{(e)}$	$\sigma_1^{(h)}$	$\sigma_1^{(e)}$
1	0.923076	0.997568	1.0	0.997845
2	0.772748	0.993896	0.993141	0.995828
3	0.615745	0.990413	0.962234	0.995027
4	0.477991	0.987729	0.904588	0.995021
5	0.366308	0.986069	0.827910	0.995403
6	0.279414	0.985355	0.742183	0.995926
7	0.213273	0.985370	0.655608	0.996463

Table 2 Quality measurer value for different support functions of the function given in (29)

$s_i(x_i)$	2^{x_i}	2^{3x_i}	2^{5x_i}	2^{8x_i}	2^{10x_i}
$\sigma_0^{(e)}$	0.0072	0.1406	0.4959	0.9282	1.0
$\sigma_1^{(e)}$	0.0628	0.4669	0.8649	0.9977	1.0

functions for each case. As an example, here we only give the structure of the support functions for $m = 4$.

$$s_i(x_i) = 0.03720997059x_i^4 + 0.2232598235x_i^3 + 0.5581495588x_i^2 + 0.6697794706x_i + 0.3200057471, \quad 1 \le i \le 4 \tag{28}$$

Using these support functions in the EMPR method we can represent the given multivariate function in terms of the constant and the univariate EMPR components. For simplicity, we do not give these structures here, instead we give Table 1 for the values of the EMPR and HDMR truncation quality measurers for the constant and univariate truncations of the function given in (26) for the different m values. It is easily seen that as the dominancy of the multiplicativity nature increases the quality measurer value obtained for the EMPR method becomes better than the value obtained for the HDMR method. If the value of m increases the univariate EMPR approximant represents the given function better than the constant approximant and this tendency can be easily observed by looking at the first order quality measurer values. These values become very close to 1 such as 0.996463 which is given for $m = 7$ and $\sigma_1^{(e)}$.

The following function is chosen as the third example to show the role of the support functions more clearly.

$$f(x_1, x_2, x_3, x_4, x_5) = 2^{10(x_1+x_2+x_3+x_4+x_5)} \tag{29}$$

In this numerical implementation, the given multivariate function is represented through the EMPR method by using several support function definition and the obtained results for the quality measurers can be examined in Table 2. As the structure of the support functions become very closer to the structure of the original function, the quality of the representation obtained through the EMPR method gets better. This shows us the importance of the support function selection in our method.

Having some observation results we may develop certain practical ways to find best support functions for maximum univariance dominancy in EMPR. The following example uses the square normalized factors of the given target function of EMPR as its support functions. For this purpose, the testing function and the corresponding support functions are defined as

$$f(x_1, x_2, x_3) = 3x_1^6 e^{x_2} \sin(x_3) \tag{30}$$

and

$$s_1(x_1) = 1.7320508x_1, \quad s_2(x_2) = 0.55949556e^{x_2}, \quad s_3(x_3) = 1.915035 \sin(x_3) \tag{31}$$

respectively.

The truncation quality measurers of the zeroth order and the first order EMPR truncations for this case are evaluated as

$$\begin{aligned} \sigma_0^{(e)} &= 0.609375 \\ \sigma_1^{(e)} &= 1.0 \end{aligned} \tag{32}$$

respectively. These measurer values show that when we match each factor to the appropriate support function, we obtain the exact representation of the given function through the EMPR method. On the other hand, if we use the cyclic permutation of the same support functions this selection changes the situation very dramatically. For instance, the following support functions can be defined

$$s_1(x_1) = 1.9150035 \sin(x_1), \quad s_2(x_2) = 1.73205080x_2, \quad s_3(x_3) = 0.55949556e^{x_3} \tag{33}$$

and the EMPR method can be used to represent the function under the effects of these support functions. As a result, the truncation quality measurers for the EMPR truncations are determined as follows.

$$\begin{aligned} \sigma_0^{(e)} &= 0.50508556 \\ \sigma_1^{(e)} &= 0.95120915 \end{aligned} \tag{34}$$

These results show us the importance of the support function selection process. The appropriate support function selection increases the quality of the EMPR method in the representation of the given multivariate function in terms of at most univariate terms.

5 Conclusion

In this work, we discussed the role of the support functions on the efficiency of the EMPR method. Several numerical implementations were designed to show the quality of the representations obtained through the EMPR method under the support function effects.

The EMPR method is a divide-and-conquer method like HDMR. Actually, HDMR is a particular case of EMPR, that is, if the value of the support functions appearing in the EMPR expansion are taken as 1, then the expansion becomes exactly same as the HDMR expansion. This means, the support function existence in the expansion allows us to have the flexibility of getting better representations than the HDMR method in multivariate problems.

Depending on the structure of the given function and the defined structure of each support function, the best (actually the exact) representation through the EMPR method can be obtained even at the zeroth order truncation. By this way, the mathematical and the computational complexities decrease with the help of the support functions.

As a result, it is clear that the selection procedure of the support functions is very important to obtain approximations in sufficiently good quality through the method. It can be easily noticed that as the structure of the support functions begins to be very similar to the original function, the performance of the method increases rapidly. On the other hand, one may want to construct a general optimization algorithm for the support function selection process. This subject may be pointed as future work.

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